Gap-planar Graphs*

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Abstract

We introduce the family of *k-gap-planar graphs* for $k \geq 0$, i.e., graphs that have a drawing in which each crossing is assigned to one of the two involved edges and each edge is assigned at most k of its crossings. This definition is motivated by applications in edge casing, as a k-gap-planar graph can be drawn crossing-free after introducing at most k local gaps per edge. We present results on the maximum density of k-gap-planar graphs, their relationship to other classes of beyond-planar graphs, characterization of k-gap-planar complete graphs, and the computational complexity of recognizing k-gap-planar graphs.

Keywords: Beyond planarity k-gap-planar graphs, Density results, Complete graphs, Recognition problem, k-planar graphs, k-quasiplanar graphs

1 Introduction

Minimizing the overall number of edge crossings in a drawing has been the main objective of a large body of literature concerning the design of algorithms to automatically draw a graph. In fact, several graph drawing algorithms assume the input graph to be planar or planarized (that is, crossings are replaced with dummy vertices which are removed in a post-processing step). More recently, cognitive experiments suggested that the absence of specific kinds of edge crossing configurations has a positive impact on the human understanding of a graph drawing [39]. These practical findings motivated a line of research, commonly called *beyond planarity*, whose focus is on non-planar graphs that can be drawn by locally avoiding specific edge crossing configurations or by guaranteeing specific properties for the edge crossings (see, e.g., [12, 36, 38, 44]).

Among the most investigated families of beyond-planar graphs are: *k-planar graphs* (see, e.g., [13, 42, 46]), which can be drawn with at most k crossings per edge; *k-quasiplanar graphs* (see, e.g., [2, 3, 27]), which can be drawn with no k pairwise crossing edges; *fan-planar graphs* (see, e.g., [10, 14, 40]), which can be drawn such that each edge is crossed by a (possibly empty) set of edges that have a common endpoint on one side; *RAC graphs* (refer, e.g., to [21]), which admit a straight-line drawing with right-angle crossings.

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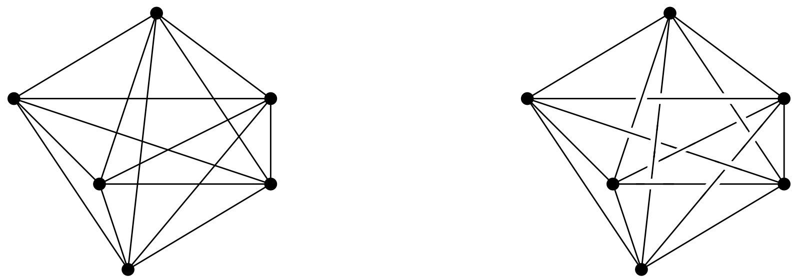


Figure 1: A drawing of a graph G (left) and its cased version where each edge is interrupted at most twice, i.e., a 2-gap-planar drawing of G (right).

In this paper we introduce a family that generalizes k-planar graphs by introducing a nonsymmetric constraint on the intersection pattern of the edges. Intuitively speaking, we charge each crossing to only one of the two edges involved in the crossing and do not allow an edge to be charged many times. This constraint is motivated by *edge casing*, a method commonly used to alleviate the visual clutter generated by crossing lines in a diagram [5, 26]. In a *cased drawing* of a graph, each crossing is resolved by locally interrupting one of the two crossing edges; see Figure 1 for an illustration. This edge casing makes only *one* of the edges involved in the crossing hard to follow whereas the other one is unaffected. Regardless of the number of crossings, the drawing will remain clear as long as no edge is cased many times; thus, an edge could participate in arbitrarily many crossings as long as the other edges are cased. Eppstein et al. [26] studied several optimization problems related to edge casing, assuming the input is a graph together with a fixed drawing. In particular, the problem of minimizing the maximum number of gaps per edge in a drawing can be solved in polynomial time (see also Section 2). We also note that a similar drawing paradigm is used by *partial edge drawings (PEDs)*, in which the central part of each edge is erased, while the two remaining stubs are required to be crossing-free (see, e.g., [17, 18]).

We formalize this idea with the family of *k-gap-planar graphs*, a family of graphs that can be drawn in the plane such that each crossing is assigned to one of the two involved edges and each edge is assigned at most k crossings (for some constant k). We present a rich set of results for k-gap-planar graphs related to classic research questions, such as bounds on the maximum density, drawability of complete graphs, complexity of the recognition problem, and relationships with other families of beyond-planar graphs. Our results can be summarized as follows:

- Every k-gap-planar graph with n vertices has $O(\sqrt{k} \cdot n)$ edges (Section 3). If $k = 1$, we prove an upper bound of $5n - 10$ for the number of edges in a 1-gap-planar multigraph with n vertices (without homotopic parallel edges), and construct 1-gap-planar (simple) graphs that attain this bound for all $n \geq 20$. Note that the same density bound is known to be tight for 2-planar graphs [46].
- We study relationships between the class of k-gap-planar graphs and other classes of beyond-planar graphs. For all $k \geq 1$, the class of $2k$-planar graphs is properly contained in the class of k-gap-planar graphs, which in turn is properly contained in the $(2k + 2)$-quasiplanar graphs (Section 4). We note that k-planar graphs are known to be $(k + 1)$-quasiplanar [4, 34]. Furthermore, we investigate the relationship between k-gap-planar graphs and d-degenerate crossing graphs, a class of graphs recently introduced by Eppstein and Gupta [25].
- The complete graph K_n is 1-gap-planar if and only if $n \leq 8$ (Section 5).
- Deciding whether a graph is 1-gap-planar is NP-complete, even when the drawing of a given graph is restricted to a fixed rotation system that is part of the input (Section 6). Note that analogous recognition problems for other families of beyond-planar graphs are also NP-hard (see, e.g., [7, 10, 14, 15, 31, 43]), while polynomial algorithms are known in some restricted settings (see, e.g., [6, 10, 15, 20, 24, 37, 35]).

Preliminaries and basic results are in Section 2. Conclusions and open problems are discussed in Section 7.

2 Preliminaries and basic results

A *drawing* Γ of a graph $G = (V, E)$ is a mapping of the vertices of V to distinct points, and of the edges of E to a continuous arcs connecting their corresponding endpoints such that no edge (arc) passes through any vertex, if two edges have a common interior point in Γ, then they cross transversely at that point, and no three edges cross at the same point. For a subset $E' \subseteq E$, the restriction of Γ to the curves representing the edges of E' is denoted by $\Gamma[E']$. A drawing Γ is *planar* if no two edges cross. A graph is *planar* if it admits a planar drawing. A *planar embedding* of a planar graph G is an equivalence class of topologically equivalent (i.e., isotopic) planar drawings of G. A *plane graph* is a planar graph with a planar embedding. A planar drawing subdivides the plane into topologically connected regions, called *faces*. The unbounded region is the *outer face*.

The *crossing number* $\mathrm{cr}(G)$ of a graph G is the smallest number of edge crossings over all drawings of G. The *crossing graph* $C(\Gamma)$ of a drawing Γ is the graph having a vertex v_e for each edge e of G, and an edge (v_e, v_f) if and only if edges e and f cross in Γ. The *planarization* Γ^* of Γ is the plane graph formed from Γ by inserting a *dummy vertex* at each crossing, and subdividing both edges with the dummy vertex. To avoid ambiguities, we call *real vertices* the vertices of Γ^* that are in V (i.e., that are not dummy).

A class of graphs is informally called "beyond-planar" if the graphs in this family admit drawings in which the intersection patterns of the edges are characterized by some forbidden configuration (see, e.g., [36, 38, 44]). Research on such graph classes is attracting increasing attention in graph theory, graph algorithms, graph drawing, and computational geometry, as these graphs represent a natural generalization of planar graphs, and their study can provide significant insights for the design of effective methods to visualize real-world networks. Indeed, the motivation for this line of research stems from both the interest raised by the combinatorial and geometric properties of these graphs, and experiments showing how the absence of particular edge crossing patterns has a positive impact on the readability of a graph drawing [39].

Among the investigated families of beyond-planar graphs are: *k-planar graphs* (see, e.g., [13, 42, 46]), which can be drawn in the plane with at most k crossings per edge; *k-quasiplanar graphs* (see, e.g., [2, 3, 27]), which can drawn without k pairwise crossing edges; *fan-planar graphs* (see, e.g., [10, 14, 40]), which can be drawn such that no edge crosses two independent edges; *fan-crossing-free graphs* [19], which can be drawn such that no edge crosses any two edges that are adjacent to each other; *planarly-connected graphs* [1], which can be drawn such that each pair of crossing edges is independent and there is a crossing-free edge that connects their endpoints; *RAC graphs* (refer, e.g., to [21]), which admit a straight-line (or polyline with few bends) drawing where any two crossing edges are perpendicular to each other.

Eppstein et al. [26] studied several optimization problems related to edge casing, assuming the input is a graph together with a fixed drawing. In particular, the problem of minimizing the maximum number of gaps per edge in a drawing can be solved in polynomial time (see also Section 2). We also note that a similar drawing paradigm is used by *partial edge drawings (PEDs)*, in which the central part of each edge is erased, while the two remaining stubs are required to be crossing-free (see, e.g., [17, 18]).

Let Γ be a drawing of a graph G. Recall that exactly two edges of G cross in one point p of Γ, and we say that these two edges are *responsible* for p. A *k-gap assignment* of Γ maps each crossing point of Γ to one of its two responsible edges so that each edge is assigned with at most k of its crossings; see, e.g., Fig. 1(right). A *gap* of an edge is a crossing assigned to it. An edge with at least one gap is *gapped*, else it is *gap-free*. A drawing is *k-gap-planar* if it admits a k-gap assignment. A graph is *k-gap-planar* if it has a k-gap-planar drawing. Note that a graph is planar if and only if it is 0-gap-planar, and that k-gap-planarity is a monotone property: every subgraph of a k-gap-planar graph is k-gap-planar. The summation of the number of gaps over all edges in a set $E' \subset E$ yields the following.

Property 1. *Let Γ be a k-gap-planar drawing of a graph $G = (V, E)$. For every $E' \subseteq E$, the subdrawing $\Gamma[E']$ contains at most $k \cdot |E'|$ crossings.*

In fact, the converse of Property 1 also holds, and we obtain the following stronger result.

Theorem 2. *Let Γ be a drawing of a graph $G = (V, E)$. The drawing Γ is k-gap-planar if and only if for each edge set $E' \subseteq E$ the subdrawing $\Gamma[E']$ contains at most $k \cdot |E'|$ crossings.*

Proof. Property 1 is the only-if direction. It remains to prove the forward direction. Let A denote the set of crossings in Γ. Further let $B = \{e_1, \dots, e_k : e \in E\}$ denote a set that consists of k copies of each edge in G. Let H be the bipartite graph whose vertex set is $A \dot\cup B$ and where a crossing $p \in A$ is connected to all copies of edges that are responsible for the crossing p.

Clearly, k-gap-planar assignments correspond bijectively to matchings M in H such that each crossing $a \in A$ is incident to an edge in M. By Hall's theorem, the bipartite graph H has a matching from A into B if and only if for each set $X \subseteq A$, we have $|N(X)| \geq |X|$.

Let $X \subseteq A$ be some subset of crossings. Let $E(X)$ denote the set of edges that are responsible for crossings in X. By considering the subdrawing $\Gamma[E(X)]$, we find $|X| \leq k|E(X)|$. Moreover, by construction of H the neighborhood $N(X)$ of X contains exactly k vertices for each edge in $E(X)$, i.e., $|N(X)| = k|E(X)|$. Thus it is $|X| \leq |N(X)|$, which is Hall's condition. Thus a k-gap-planar assignment exists, showing that Γ is k-gap-planar. □

Note: David Wood (personal communication) has suggested an alternative proof of the above statement: Hakimi [33] proved that a graph has an orientation with maximum outdegree at most k if and only if every subgraph has average degree at most $2k$. Theorem 2 immediately follows by applying this result to the intersection graph of the edges in a drawing of a graph.

A k-gap assignment of a drawing Γ corresponds to orienting the edges of the crossing graph $C(\Gamma)$ such that each vertex has indegree at most k (intuitively, orienting a crossing towards an edge means we assign the crossing to that edge). Since finding an orientation of a graph with the smallest maximum indegree corresponds to finding its pseudoarboricity [28, 47], Property 3 below follows. A *pseudoforest* is a graph in which every connected component has at most one cycle, and the *pseudoarboricity* of a graph is the smallest number of pseudoforests needed to cover all its edges.

Property 3. *A graph is k-gap-planar if and only if it admits a drawing whose crossing graph has pseudoarboricity at most k.*

Given a drawing Γ of a graph $G = (V, E)$, we can find the minimum $k \geq 0$ such that Γ is k-gap-planar in $O(|E|^4)$ time, due to the fact that one can find an orientation of $C(\Gamma)$ with the smallest maximum indegree in time quadratic in the number of edges of $C(\Gamma)$ [49].

Note: In an earlier versions of this paper [8, 9] we gave an upper bound on the treewidth of k-gap-planar graphs. Our bounds were based on a result by Dujmović, Eppstein, and Wood [23] that bounded the treewidth of a graph as a function of the number of vertices and the crossing number. Unfortunately, their result turned out to be incorrect [**?**]. Thus, it still remains open to show that k-gap-planar graphs have $O(f(k)\sqrt{n})$ treewidth for some function f.

3 Density of k-gap-planar graphs

We begin with an upper bound on the number of edges of k-gap-planar graphs.

Theorem 4. *A k-gap-planar graph on $n \geq 3$ vertices has $O(\sqrt{k} \cdot n)$ edges.*

Proof. The crossing number of a graph G with n vertices and m edges is bounded by $\mathrm{cr}(G) \geq \frac{1024}{31827} \cdot m^3/n^2$ when $m \geq \frac{103}{6}n$ [45]. Combined with the bound $\mathrm{cr}(G) \leq k \cdot m$ (Property 1), we obtain

$$\frac{1024}{31827} \cdot \frac{m^3}{n^2} \leq \mathrm{cr}(G) \leq km,$$

which implies $m \leq \max(5.58\sqrt{k}, 17.17) \cdot n$, as required. □

Better upper bounds are possible for small values of k, in particular for $k = 1$. Pach et al. [45] proved that a graph G with $n \geq 3$ vertices satisfies $\mathrm{cr}(G) \geq \frac{7}{3}m - \frac{25}{3}(n-2)$. Combined with the bound $\mathrm{cr}(G) \leq k \cdot m$, we have

$$m \leq \frac{25(n-2)}{7-3k}.$$

For $k = 1$ (i.e., for 1-gap-planar graphs), this gives $m \leq 6.25n - 12.5$. We now show how to improve this bound to $m \leq 5n - 10$ (see Theorem 6 below). The idea is to follow a strategy developed by Pach and Tóth [46] and Bekos et al. [13] on the density of 2- and 3-planar graphs, with several important differences.

In order to accommodate the elementary operations in the proof of Theorem 6, we work on a broader class of graphs. A drawing Γ of a multigraph $G = (V, E)$ is k-gap-planar if it admits a k-gap assignment and no two parallel edges are homotopic. A multigraph is *k-gap-planar* if it has a k-gap-planar drawing. Two parallel edges $e_1 = e_2 = (u, v)$ are *homotopic* in a drawing Γ, if the drawings are continuous arcs $\gamma_1 : [0, 1] \to \mathbb{R}^2$ and $\gamma_1 : [0, 1]\mathbb{R}^2$, and there is a continuous function $h : [0, 1]^2 \to \mathbb{R}^2$ such that $\gamma_1(t) = h(0, t)$, $\gamma_2(t) = h(1, t)$, $\Gamma(u) = h(t, 0)$, and $\Gamma(v) = h(t, 1)$ for all $t \in [0, 1]$, and h does not map any point of the open square $(0, 1)^2$ to a vertex in Γ. Intuitively, γ_1 can be continuously deformed into γ_2 with fixed endpoints and without passing through any vertex in Γ. In particular, in a 1-gap-planar drawing, two homotopic parallel edges either cross no other edge (they might cross each other), or they both cross the same edge.

Let $n \in \mathbb{N}$, $n \geq 3$. Let $G = (V, E)$ be a 1-gap-planar multigraph with n vertices that has the maximum number of edges possible over all n-vertex 1-gap-planar multigraphs; and let Γ be a 1-gap-planar drawing of G. Let $H = (V, E')$ be a sub-multigraph of G, where $E' \subseteq E$ is a maximum multiset of edges that are pairwise noncrossing in $\Gamma[E']$, and if there are several such sub-mustligraphs, then H has the fewest connected components.

We first show that $H = (V, E')$ is a *triangulation*, that is, a plane multigraph in which every face is bounded by a walk with three vertices and three edges.

Lemma 5. *The multigraph H is a triangulation.*

The proof of Lemma 5 is deferred to Section 3.1. We can now show that $|E| \leq 5n - 10$. We state a stronger result (for multigraphs) that immediately implies the same density bound for simple graphs (Corollary 7).

Theorem 6. *A multigraph on $n \geq 3$ vertices that has a 1-gap-planar drawing in which no two parallel edges are homotopic has at most $5n - 10$ edges.*

Corollary 7. *A 1-gap-planar (simple) graph on $n \geq 3$ vertices has at most $5n - 10$ edges.*

Proof of Theorem 6. By Lemma 5, $H = (V, E')$ is a triangulation. By Euler's polyhedron theorem, it has $3n - 6$ edges and $2n - 4$ triangular faces. Consider the edges in $E'' = E \setminus E'$. It remains to show that $|E''| \leq 2n - 4$.

The embedding of edge $e \in E''$ is a Jordan arc that visits two or more triangle faces of H. We call the first and last triangles along e the *end triangles* of e. For an end triangle Δ, the connected component of $e \cap \Delta$ incident to a vertex of Δ is called an *end portion*. We use the following charging scheme.

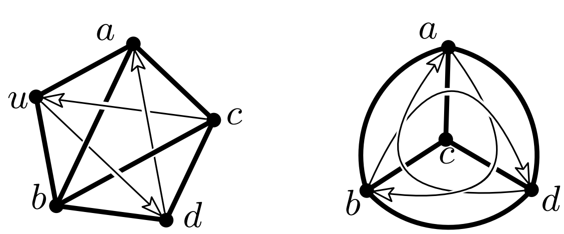


Figure 2: Example for the charging scheme in the proof of Theorem 6. Bold edges are in a crossing-free triangulation $H = (V, E')$. Every edge in $E \setminus E'$ is charged to a triangle of H as indicated by arrows. Left: a simple graph where edge (c, u) is charged to Δabu, edge (d, u) to Δacd, and edge (a, d) to Δabc. Right: A 1-gap-planar multigraph with nonhomotopic parallel edges.

Each edge $e \in E''$ charges one unit to a triangle face of H as follows. (Refer to Fig. 2.) If e has an end portion that has a gap neither in the interior nor on the boundary of the corresponding end triangle Δ, then e charges one unit to Δ. (If neither end portions of e has a gap in the interior or on the boundary of its end triangle, then e charges one arbitrary end triangle.) Otherwise the two end portions of e lie in two adjacent triangles, say, Δ_1 and Δ_2, and e uses its gap to cross an edge e' on the boundary between Δ_1 and Δ_2; in this case e charges one unit to Δ_1 or Δ_2 as follows: If e' has a gap and the edge $e'' \in E''$ passing through this gap charges Δ_1 (because e'' has an end portion $e'' \cap \Delta_1$ that has a gap neither in the interior nor on the boundary of Δ_1), then e charges Δ_2, otherwise it charges Δ_1.

We claim that each face of H receives at most one unit of charge. Let $\Delta = \Delta abc$ be a face in H. Suppose to the contrary that Δ receives positive charge from two edges, say $e_1, e_2 \in E''$. Then both edges have an end portion in Δ that do not have gaps in the interior of Δ. Consequently, the end portions of e_1 and e_2 in Δ cannot cross, and so they are incident to the same vertex of Δ. Therefore, the both end portions $e_1 \cap \Delta$ and $e_2 \cap \Delta$ are incident to the same vertex of Δ, say a, and cross the edge of Δ opposite to a, namely (b, c). Let $\Delta' = \Delta' bcd$ be the face of the plane graph H on the opposite side of (b, c).

Assume first that the end portion $e_1 \cap \Delta$ has a gap neither in the interior nor on the boundary of Δ. Then e_1 passes through the gap of (b, c). Since (b, c) has at most one gap in a 1-gap-planar drawing, e_2 uses its own gap to cross (b, c). By our charging scheme, this implies that $e_2 = (a, d)$, and it must charges one unit to Δ' (rather than Δ). Next assume that (b, c) does not have any gap. Then e_1 and e_2 each use their own gaps to cross (b, c). Both e_1 and e_2 are homotopic to an edge (a, d) lying in $\Delta_1 \cup \Delta_2$ by our charging scheme. All cases lead to a contradiction, hence Δ receives at most 1 unit of charge, as claimed. Consequently, $|E''|$ is bounded above by the number of faces of H, which is $2n - 4$, as required. ☐

3.1 Proof of Lemma 5

We start with a few basic observations. Let G be an edge-maximal multigraph on n vertices that has a 1-gap-planar drawing without homtopic parallel edges.

Lemma 8. *Graph $G = (V, E)$ is connected.*

Proof. Suppose, to the contrary, that G is disconnected. Let $G_1 = (V_1, E_1)$ be one component, and let $G_2 = (V_2, E_2)$ be the disjoint union of all other components (i.e., $V_2 = V \setminus V_1$ and $E_2 = E \setminus E_1$). For $i = 1, 2$, let $\Gamma_i = \Gamma[E_i]$ (i.e., the drawing of G_i inherited from G), and let Γ_i^* be the planarization of Γ_i.

Let f_2 be a face in Γ_2^* incident to some vertex $v_2 \in V_2$. Apply a projective transformation to Γ_1 so that the outer face is incident to some vertex $v_1 \in V_1$; followed by an affine transformation that maps Γ_1 into the interior of face f_2. We obtain a 1-gap-planar drawing of G in which we can insert a new crossing-free edge (v_1, v_2), between two distinct components of G, contradicting the maximality of G. ☐

Recall that Γ is a 1-gap-planar drawing of G with the minimum number of crossings. We show that this implies that Γ is a *simple topological* drawing, that is, no edge crosses itself and every pair of edges cross at most once. This follows from standard simplification techniques, but we provide the proof for completeness.

Lemma 9. *Γ is a simple topological drawing.*

Proof. Suppose the drawing γ_0 of an edge $e_0 = (u, v)$ crosses itself at point c_0. Then γ_0 crosses itself only once, and this crossing is charged to edge e, hence all other crossings of γ_0 are charged to other edges. We can redraw e by eliminating the loop of γ_0. This yields a new 1-gap-planar drawing of G with at least one fewer crossings, contradicting the minimality of Γ.

Suppose the drawings γ_1 and γ_2 of edges e_1 and e_2 cross at points c_1 and c_2. Then they cross exactly twice and these two crossings are charged to e_1 and e_2, hence any other crossing of e_1 or e_2 with some edge e_3 is charged to e_3. We can redraw e_1 and e_2 in $\gamma_1 \cup \gamma_2$ by exchanging their subarcs between c_1 and c_2 such that both crossings are eliminated. This yields a new 1-gap-planar drawing of G with fewer crossings, contradicting the minimality of Γ. ☐

Since G is connected, every face in the planarization Γ^* of Γ has a connected boundary. The *boundary walk* of a face f is a closed walk $(a_1, a_2, \ldots, a_m)$ in Γ^* such that f lies on the left hand side of each edge (a_i, a_{i+1}) along the walk; and every two consecutive edges of the walk, (a_{i-1}, a_i) and (a_i, a_{i+1}), are also consecutive in the counterclockwise rotation of all edges incident to a_i. Let F_0 denote the set of faces in the planarization Γ^* that are not incident to any vertex in V.

Lemma 10. *If $f \in F_0$, then the boundary walk of f is*

1. *a simple cycle (i.e., has no repeated vertices) with at least 3 vertices;*

2. *disjoint from the boundary walk of any other face in F_0.*

Proof. 1. Let $f \in F_0$, and let $w = (a_1, a_2, \ldots, a_\ell)$ be its boundary walk. By Lemma 9, we have $\ell \geq 3$. Let $C_f = \{a_1, \ldots, a_\ell\}$ be the set of vertices in w; and let $E_f \subset E$ be the set of edges in G that contain some edge of w. It suffices to show that $|C_f| = \ell$, and then w has no repeated vertices, hence it is a simple cycle.

Suppose, to the contrary, that the vertices in w are not distinct. Since $f \in F_0$, all vertices in w are crossings in the drawing Γ, consequently they all have degree 4 in the planarization Γ^*. If $a_i = a_j$, $i \neq j$, then a_i and a_j cannot be consecutive vertices in w, and two pairs of edges from (a_{i-1}, a_i), (a_i, a_{i+1}), (a_{j-1}, a_j), (a_j, a_{j+1}) are part of the same edge in E. If $|C_f| = \ell - k$, for some $k \in \mathbb{N}$, then $|E_f| \leq \ell - 2k$. This implies $|E_f| < |C_f|$. That is, the edges in E_f are involved in more than $|E_f|$ crossings, contradicting the assumption that Γ is a 1-gap-planar drawing.

2. Let $f_1, f_2 \in F_0$ be two faces, with boundary walks $w_1 = (a_1, \ldots, a_\ell)$ and $w_2 = (b_1, \ldots, b_{\ell'})$. Both w_1 and w_2 are simple cycles by part 1. For $i = 1, 2$, let C_i be the set of vertices in w_i, and $E_i \subseteq E$ the set of edges of G that contain the edges of the walk w_i.

Note that w_1 and w_2 cannot share two consecutive edges, say (a_{i-1}, a_i) and (a_i, a_{i+1}), since the middle vertex a_i has degree 4 in Γ^*. When w_1 and w_2 have a common edge, say $(a_i, a_{i+1}) = (b_{j+1}, b_j)$, then three pairs of edges from (a_{i-1}, a_i), (a_i, a_{i+1}), (a_{i+1}, a_{i+2}), (b_{j-1}, b_j), (b_j, b_{j+1}) (b_{j+1}, b_{j+2}) are part of the same edge in E. When w_1 and w_2 have a common vertex $a_i = b_j$ but no common edge incident to $a_i = b_j$, then two pairs of edges from (a_{i-1}, a_i), (a_i, a_{i+1}), (b_{j-1}, b_j), (b_j, b_{j+1}) are part of the same edge in E. This implies $|E_1 \cup E_2| < |C_1 \cup C_2|$. That is, the edges in $E_1 \cup E_2$ are involved in more than $|E_1 \cup E_2|$ crossings, contradicting the assumption that Γ is 1-gap-planar. □

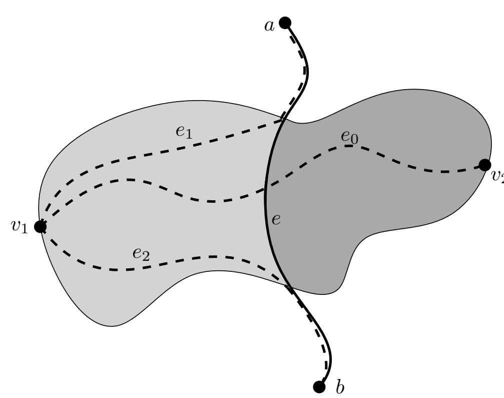


Figure 3: Illustration for the proof of Lemma 11: Two adjacent faces in the planarization Γ^* that are incident to two distinct vertices, v_1 and v_2, separated by an edge $e = (a, b)$. If we replace the edge e by either $e_0 = (v_1, v_2)$ or both $e_1 = (v_1, a)$ and $e_2 = (v_1, b)$, we obtain a 1-gap-planar drawing of a graph that has either one fewer component in H or one more edge.

Recall that $H = (V, E')$ is a sub-multigraph of G, where $E' \subseteq E$ is a maximum multiset of edges that are pairwise noncrossing in $\Gamma[E']$, and if there are several such sub-mustligraphs, then H has the fewest connected components.

Lemma 11. *Graph $H = (V, E')$ is connected.*

Proof. Suppose, to the contrary, that H is disconnected. Let $H_1 = (V_1, E'_1)$ be one component, and let $H_2 = (V_2, E'_2)$, where $V_2 = V \setminus V_1$ and $E'_2 = E' \setminus E'_1$.

Consider the faces in the planarization Γ^* of Γ. Notice that there is no face in Γ^* incident to a vertex $v_1 \in V_1$ and a vertex $v_2 \in V_2$, otherwise we could either add a new edge (v_1, v_2) contradicting the maximality of G, or redraw an existing edge (v_1, v_2) to pass through the interior of this face, contradicting the maximality of E'.

Consequently, we can partition the faces in Γ^* into three categories: For $i = 1, 2$, let F_i be the set of faces incident to a vertex in V_i; and let F_0 be the set of faces incident to neither V_1 nor V_2. By Lemma 10, the region obtained by removing all faces in F_0 (i.e., $\mathbb{R}^2 \setminus \bigcup_{f \in F_0}$) is connected. Consequently, there exist some faces $f_1 \in F_1$ and $f_2 \in F_2$ that have a common edge in Γ^*. Let $v_1 \in V_1$ and $v_2 \in V_2$ be incident to $f_1 \in F_1$ and $f_2 \in F_2$. Let $e \in E$ be the edge on the common boundary of f_1 and f_2, and denote its endpoints by $a, b \in V$.

We consider three possible edges that we describe together with their drawings (up to homotopy equivalence) with respect to Γ: Let $e_0 = (v_1, v_2)$ be an edge such that it lies in $f_1 \cup f_2$; let $e_1 = (v_1, a)$ (resp., $e_2 = (v_1, b)$) be an edge such that it starts in f_1 and closely follows edge e from f_1 to its endpoint a (resp., b). Refer to Fig. 3. (If edge e_0 (resp., e_1 or e_2) is homotopic to an existing edge in Γ, then we can redraw it as described above, and maintain a 1-gap-planar drawing of G).

Note that $e_0 \in E$, otherwise we can add e_0 to E with the drawing described above, and charge the crossing $e_0 \cap e$ to e_0, contradicting the maximality of G. Note also that e_1 and e_2 (which may or may not be present in G) form a path between a and b. We distinguish two cases:

- Assume $e \notin E'$. We can add e_0 to E', contradicting the maximality of E'.
- Assume $e \in E'$. If neither e_1 nor e_2 is present in G and Γ, then we can modify E by replacing e with these edges, contradicting the maximality of E. If both e_1 and e_2 are present in G, then they both are in E' by the maximality of E'. In this case, we can modify E' by replacing e with e_0. Then a, b, v_1, and v_2 will be in the same component of H, contradicting the tie-breaking rule that H was a maximum crossing-free subgraph with the fewest components. Otherwise we can modify both E and E' by replacing e with e_1 or e_2 (whichever is not already present in Γ), and then add edge e_0 to E', which contradicts the maximality of H.

All cases lead to a contradiction, which completes the proof. $\square$

In the proof of Lemma 5, we shall use Sperner's Lemma [48], a well-known discrete analogue of Brouwer's fixed point theorem.

Lemma 12. (Sperner [48]) *Let K be a geometric simplicial complex in the plane, where the union of faces is homeomorphic to a disk. Assume that each vertex is assigned a color from the set $\{1, 2, 3\}$ such that three vertices $v_1, v_2, v_3 \in \partial K$ are colored 1, 2, and 3, respectively, and for any pair $i, j \in \{1, 2, 3\}$, the vertices on the path between v_i and v_j along ∂K that does not contain the 3rd vertex are colored with $\{i, j\}$. Then K contains a triangle whose vertices have all three different colors.*

We are now ready to prove Lemma 5, restated in the following form.

Lemma 13. *The multigraph H is a triangulation, that is, a plane multi-graph in which every face is bounded by a walk with three vertices and three edges.*

Proof. Suppose, to the contrary, that H is not a triangulation. Then H has a face f whose boundary walk $w = (v_1, v_2, \ldots, v_m)$ has more than three vertices (i.e., $m \geq 4$). To simplify notation, we assume that w is a simple cycle; this assumption is not essential for the proof.

Let P_f be the subgraph of Γ^* formed by all edges and vertices lying in the interior or on the boundary of f; let V_f denote the set of vertices of P_f (it consists of $v_1, \ldots, v_m$ and all crossings in the interior or on the boundary of f); and let F denote the set of faces of Γ^* that lie in f. Let $F_0 \subseteq F$ be the set of faces that are not incident to any vertex in $\{v_1, \ldots, v_m\}$; and for $i = 1, \ldots, m$, let $F_i \subseteq F$ be the set of faces incident to v_i.

We note the following properties of the arrangement of faces in F.

(P1) A face $f_i \in F_i$ cannot be incident to a vertex v_j, $j \notin \{i-1, i, i+1\}$. Indeed, otherwise we could add a new edge $e = (v_i, v_j)$ to G that lies in f_i. Note that Γ does not contain a homotopic parallel edge, otherwise it would lie in the face f, and could be added to H, contradicting the maximality of H.

(P2) A face $f_i \in F_i$ cannot be adjacent to a face $f_j \in F_j$, $j \notin \{i-1, i, i+1\}$. Indeed, otherwise we can add a new edge (v_i, v_j) to G such that (v_i, v_j) lies in $f_i \cup f_j$ and uses a gap to cross the boundary between these faces (Fig. 4(a)). Again Γ cannot contain a homotopic parallel edge, otherwise it would lie in the face f, and could be added to H, contradicting the maximality of H.

(P3) A vertex $c \in V_f \setminus V$ cannot is incident to two faces $f_i \in F_i$ and $f_j \in F_j$, $j \notin \{i-1, i, i+1\}$ Suppose, to the contrary, that there is such a vertex c (Fig. 4(b)). Then two edges $e_1, e_2 \in E \setminus E'$ cross at c. We can replace edge e_1 with a new edge (v_i, v_j) that lies in $f_i \cup f_j$ and that crosses edge e_2 at c. The new edge can be inserted into both G and H, contradicting the maximality of H. In this case, Γ cannot already contain a homotopic parallel edge, otherwise it could be added to H, contradicting the maximality of H.

(P4) A face $f_0 \in F_0$ cannot be adjacent to two faces $f_i \in F_i$ and $f_j \in F_j$ $j \notin \{i-1, i, i+1\}$. Suppose to the contrary that there is such a face f_0 (Fig. 4(c)). Then two edges $e_1, e_2 \in E \setminus E'$ are on the common boundary of the adjacent pairs f_i, f_0 and f_0, f_j. We can replace edge e_1 with a new edge (v_i, v_j) that lies in $f_i \cup f_0 \cup f_j$ that crosses edge e_2. The new edge can be inserted into both G and H, contradicting the maximality of H. Again, Γ cannot already contain a homotopic parallel edge, otherwise it could be added to H, contradicting the maximality of H.

We next distinguish two cases.

Case 1. For every $i \in \{1, \ldots, m\}$, the edge (v_i, v_{i+1}) is incident to faces in $F_0 \cup F_i \cup F_{i+1}$ only. We use Sperner's Lemma [48] for a triangulation K of the *dual graph* on the faces $F_1 \cup \cdots \cup F_m$, that we define here. We first create the *standard dual graph* of F: The nodes correspond to the faces in F; and two nodes are adjacent if and only if the corresponding faces are adjacent in Γ^*. We then triangulate the standard dual graph as follows. For every crossing $c \in V_f$ in the interior of f is incident to four faces in F, and their adjacency graph forms a 4-cycle in the standard dual. By Lemma 10(2), at least three of those faces are in $F \setminus F_0$. We triangulate the 4-cycle by an arbitrary diagonal between two faces in $F \setminus F_0$. Note that the faces in F_0 still form an independent set by Lemma 10(2). We call the resulting graph the *modified dual graph* of F. By Property (P4), every face in F_0 is adjacent to at most one side of f. Consequently, the modified dual graph is 2-connected, and the neighbors each face $f_0 \in F_0$ form a cycle or a path. Finally, remove all nodes corresponding to F_0 from the modified dual graph, and triangulate the cycle or path of neighboring nodes arbitrarily to obtain a triangulation K. The condition in Case 1 implies that K is a geometric simplicial complex, where the union of faces is homeomorphic to a disk.

We now define a 3-coloring of K (the coloring need not be proper). Assign color 1 to all faces in F_1. For $i = 2, \ldots, m$, assign color 2 to all faces in $F_i \setminus \bigcup_{j<i} F_j$ if i is even, and color 3 if i is odd. Since $m \geq 4$, each of the three colors are used at least once.

We have seen that K satisfies the conditions of Sperner's Lemma. The Lemma implies that K contains a triangle whose nodes have all three different colors, say $f_i \in F_i$, $f_j \in F_j$, and $f_k \in F_k$. Without loss of generality, assume that $j \notin \{i-1, i+1\}$. (Possibly we have $k \in \{i-1, i+1\} \cap \{j-1, j+1\}$, e.g., $i = 1$, $k = 2$, and $j = 3$.) Consider three cases depending on how the edge (f_i, f_j) in K was created:

- If f_i and f_j are adjacent in Γ^*, then (P2) is violated.
- If a vertex $c \in V_f$ is incident to both f_i and f_j, then (P3) is violated.
- If a face $f_0 \in F_0$ is adjacent to both f_i and f_j, then (P4) is violated.

All three subcases lead to a contradiction.

Case 2. There is an index $i \in \{1, \ldots, m\}$ such that (v_i, v_{i+1}) is incident to a face in F_j for some $j \neq 0, i, i+1$. Without loss of generality, we may assume that edge (v_1, v_m) is incident to a face in F_j for

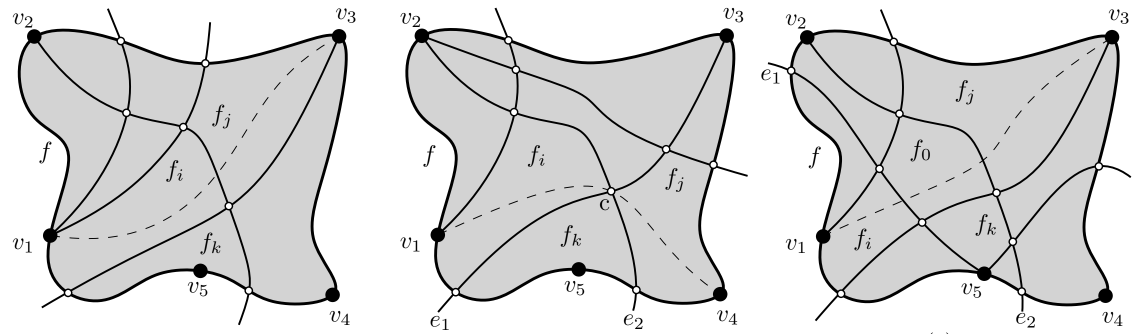


Figure 4: Illustration for the proof of Lemma 13 with $m = 5$: The dual graph K (which is not shown in the figure) has a triangle whose nodes have three different colors, say $f_i \in F_i$, $f_j \in F_j$, and $f_k \in F_k$, where $j \notin \{i-1, i+1\}$. (a) Faces f_i and f_j are adjacent. (b) Vertex $c \in V_f$ is incident to both f_i and f_j. (c) A face $f_0 \in F_0$ is adjacent to both f_i and f_j.

some $1 < j < m$. (Refer to Fig. 5 where $m = 5$.) Note that edge (v_1, v_m) must be incident to some face in F_j for *all* $1 \leq j \leq m$; otherwise (v_1, v_m) would be incident to two faces, $f_i \in F_i$ and $f_j \in F_j$, $j \notin \{i-1, i, i+1\}$, that are either adjacent to each other or both adjacent to some face $f_0 \in F_0$; and then we could add a new edge (v_i, v_j) lying in $f_i \cup f_j$ or $f_i \cup f_0 \cup f_j$.

It follows that there are faces $f_2 \in F_2$ and $f_3 \in F_3$ that are incident to some point $c \in (v_1, v_m)$ (see Fig. 5(a)); or both are adjacent to some common face $f_0 \in F_0$ that is incident to (v_1, v_m) (see Fig. 5(b)).

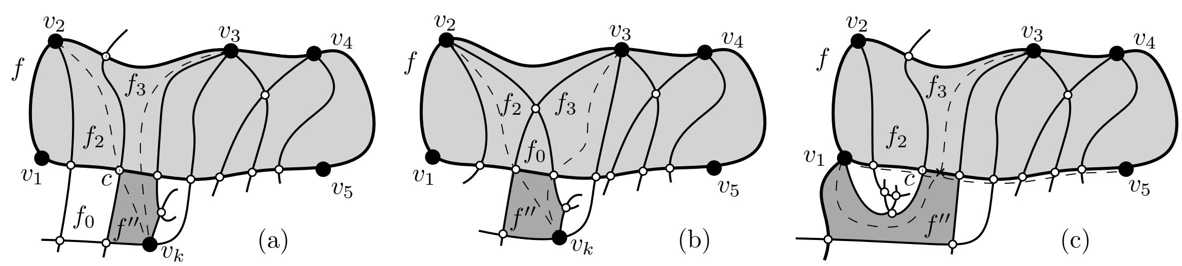
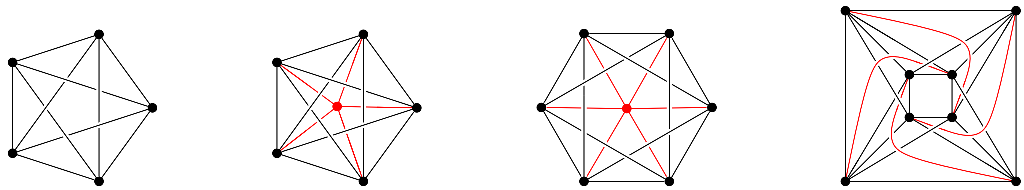


Figure 5: Illustration for the proof of Lemma 13 with $m = 5$: Edge (v_1, v_5) is incident to some face in F_j for *all* $1 \leq j \leq 5$. (a) Faces $f_2 \in F_2$ and $f_3 \in F_3$ that are incident to some point $c \in (v_1, v_m)$. (b) Faces $f_2 \in F_2$ and $f_3 \in F_3$ adjacent to face $f_0 \in F_0$ that is incident to (v_1, v_m). (c) Face f'' is incident to v_1.

Consider the face f' of H on the opposite side of (v_1, v_m), and let F' be the set of faces in the planarization Γ^* contained in f'. Let $f'' \in F'$ be a face incident to $c \in (v_1, v_m)$ or adjacent to face f_0. By Lemma 10(2), we may assume that f'' is incident to a vertex v_k on the boundary of the face f'. It is possible that $v_k = v_1$ or $v_k = v_m$.

- If $v_k \notin \{v_1, v_m\}$, we modify G, Γ, and H as follows (Fig. 5(a)–(b)): Consider the possible edges (v_2, v_k) and (v_3, v_k) that lie in $f_2 \cup f''$ and $f_3 \cup f''$, respectively, they each cross (v_1, v_m) and at most one additional edge at c or at a vertex of f_0. If (v_2, v_k) or (v_3, v_k) is present in G and Γ (as a homotopic copy), it can be redrawn to lie in $f_2 \cup f''$ and $f_3 \cup f''$, respectively. If (v_2, v_k) or (v_3, v_k) is not present in G and Γ, we then insert it and remove the edge (v_1, v_m). Finally, we can modify E' by replacing (v_1, v_m) with (v_2, v_k) and (v_3, v_k), contradicting the maximality of E'.
- If $v_k = v_1$, then we modify G, Γ, and H as follows (Fig. 5(c)): Add a new edge (v_1, v_3) that lies in $f_3 \cup f''$ or $f_3 \cup f_0 \cup f''$, and crosses (v_1, v_m) at a point x on the boundary between f'' and f_3. Then redraw the edges (v_1, v_m) and (v_1, v_3) by exchanging their initial arcs between v_1 and x, and



(a) (b) (c) (d)

Figure 6: Patterns that produce 1-gap-planar graphs with n vertices and $5n - \Theta(1)$ edges.

eliminating the crossing at x. The edge (v_1, v_3) was not previously present in G, otherwise it would be homotopic to a diagonal (v_1, v_3) of the face f of H, contradicting the maximality of E'. (However, a homotopic copy of the new drawing of edge (v_1, v_m) may be already present in Γ, in which case, the total number of edges in G remains the same). Modify E' by replacing the edge (v_1, v_m) of face f with the new edges (v_1, v_3) and (v_1, v_m) described here. This contradicts the maximality of E'.

- If $v_k = v_m$ and $v_{m-1} = v_3$, we make similar changes: We increase $|E'|$ by replacing the edge (v_1, v_m) of f with a new edge (v_2, v_m) and a new drawing of the edge (v_1, v_m).

All cases lead to a contradiction. Therefore, our initial assumption must be dropped, consequently the multigraph H is a triangulation, as claimed. $\square$

3.2 Lower bound constructions

We now show that the bound of Theorem 6 is worst-case optimal. A 2-planar graph with n vertices and $5n - 10$ edges is also 1-gap-planar (see Lemma 19). Pach and Tóth [46] construct such a graph by starting with a plane graph with pentagonal faces (e.g., using nested copies of an icosahedron), and then add all five diagonals in each pentagonal face; see Fig. 6(a). This construction yields a 1-gap-planar graph with n vertices and $m = 5n - 10$ edges for all $n \geq 20$, $n \equiv 5 \pmod{15}$.

We can modify this construction by inserting a new vertex in one or more pentagons, and connecting it to the 5 vertices of the pentagon; see Fig. 6(b). Every new edge crosses exactly one diagonal of the pentagon, so the new crossings can be charged to the new edges. Since every new vertex has degree 5, the equation $m = 5n - 10$ prevails. By inserting a suitable number of vertices into pentagons, we obtain constructions for $n \in \mathbb{N}$ such that $20 \leq n \leq 32$ or $n \geq 38$. A similar construction is based on hexagonal faces; see Fig. 6(c). Start with a *fullerene*, that is, a 3-regular, plane graph G_0 with n_0 vertices, 12 pentagon faces, and $n_0/2 - 10$ hexagon faces (including the external face). Add diagonals in each face to connect a vertex to their second neighbors (the graph is 2-planar so far); finally insert a new vertex in each face of G_0, and connect them to all vertices of that face. We obtain a 1-gap-planar graph G. The number of vertices is $n = n_0 + 12 + (n_0/2 - 10) = \frac{3}{2}n_0 + 2$, and the number of edges is $m = \frac{3}{2}n_0 + 10 \cdot 12 + 12 \cdot (n_0/2 - 10) = \frac{15}{2}n_0 = 5n - 10$. Fullerenes exist for $n_0 = 20$ and for all even integers $n_0 \geq 24$ [16]. This yields a lower bound of $5n - 10$ for $n = 32$ and for all $n \geq 35$ where $n \equiv 2 \mod 3$. However, similarly to the previous construction, the equation $m = 5n - 10$ prevails if we *delete* up to 12 vertices inserted into pentagons. Consequently, the upper bound $5n - 10$ in Theorem 6 is tight for all $n \geq 20$.

Theorem 14. *For every integer $n \geq 20$ there exists a 1-gap-planar (simple) graph with n vertices and $5n - 10$ edges.*

We mention a third, slightly weaker construction, which is based on a sequence of nested squares. Fig. 6(d) shows how to add 16 edges between two consecutive squares such that the 16 crossings are assigned to distinct edges. We can add two diagonals in the external face and the innermost square. Using s squares, we have $n = 4s$, and $m = 4s + 16(s - 1) + 2 \cdot 2 = 20s - 12 = 5n - 12$. In particular, for $s = 2$ this yields a drawing of K_8; see Fig. 7(a).

If we allow 1-gap-planar muligraphs (with nonhomotopic parallel edges in a 1-gap-planar drawing), then we can construct smaller configurations for which the upper bound $5n - 10$ of Theorem 6 is tight. Start with a regular polygon P_0 with $n_0 \geq 5$ vertices. Subdivide the interior and the exterior of P_0 independently into one pentagon and $n_0 - 5$ triangle faces using $n_0 - 5$ diagonals. In each of the two pentagons, add five edges as shown in Fig. 2(left). In each triangle, add a new vertex and six new edges as shown in Fig. 2(right). We obtain a 1-gap-planar drawing of a multigraph with $n = n_0 + 2(n_0 - 5) = 3n_0 - 10$ vertices and $n_0 + 2(n_0 - 5) + 2 \cdot 5 + 2(n_0 - 5) \cdot 6 = 15n_0 - 60 = 5n - 10$ edges for all $n \geq 5$, $n \equiv 2 \mod 3$. By inserting a new vertex in one or two pentagons, and connecting it to the 5 vertices of the pentagon as in Fig. 6(b), the lower bound extends for all integers $n \geq 5$. We summarize our lower bound for multigraphs in the following theorem.

Theorem 15. *For every integer $n \geq 5$, there exists a 1-gap-planar multigraph with n vertices and $5n - 10$ edges.*

4 Relationship between k-gap-planar graphs and other families of beyond-planar graphs

In this section we prove the following theorem.

Theorem 16. *For every integer $k \geq 1$, the following relationships hold.*

$$(2k)\text{-PLANAR} \subsetneq k\text{-GAP-PLANAR} \subsetneq (2k+2)\text{-QUASIPLANAR}$$

We begin by showing the following.

Lemma 17. *For all $k \geq 1$, every k-gap-planar drawing is $(2k+2)$-quasiplanar.*

We also need to show that for every $k \in \mathbb{N}$ there is a $(2k+2)$-quasiplanar graph that is not k-gap-planar. We prove a stronger statement:

Lemma 18. *For all $k \geq 1$, there is a 3-quasiplanar graph G_k that is not k-gap-planar.*

Proof. Let $k \in \mathbb{N}$. We construct a graph $G_k = (V, E)$ as follows. Start with $K_{3,3}$ and replace each edge by $t = 19k$ edge-disjoint paths of length 2. Note that the total number of edges is $|E| = 9 \cdot 2t = 18t$. Graph G_k is 3-quasiplanar. Since $\mathrm{cr}(K_{3,3}) = 1$, it admits a drawing with precisely one crossing. The paths of length 2 can be drawn close to the edges of $K_{3,3}$ such that two paths cross if and only if the two corresponding edges of $K_{3,3}$ cross. Consequently, G_k admits a drawing Γ_0 in which any two crossing edges are part of two paths that correspond to two crossing edges of $K_{3,3}$, which in turn implies that no three edges in Γ_0 pairwise cross.

Suppose that G_k admits a k-gap-planar drawing Γ. Then the total number of crossings is at most $k|E| = 18kt$. We derive a contradiction by showing that $\mathrm{cr}(G_k) \geq 19kt$. If we choose one of the t paths for each of the 9 edges of $K_{3,3}$ independently, then we obtain a subdivision of $K_{3,3}$, therefore there is a crossing between at least one pair of paths. There are t^9 ways to choose a path for each of the 9 edges of $K_{3,3}$. Each crossing between two paths in Γ is counted $t^{9-2} = t^7$ times. Consequently, the total number of crossings in Γ is at least $t^2 = 19kt$. □

We now show that every $(2k)$-planar drawing is k-gap-planar. We note that a similar result can be derived from [18] (Lemma 10), but only for the case $k = 1$. A bipartite graph with vertex sets A and B is denoted as $H = (A, B, E)$. A *matching* from A into B is a set $M \subseteq E$ such that every vertex in A is incident to exactly one edge in M and every vertex in B is incident to at most one edge in M. The *neighborhood* of a subset $A' \subseteq A$ is the set of all vertices in B that are adjacent to a vertex in A', and is denoted as $N(A')$.

Lemma 19. *For all $k \geq 1$, every $(2k)$-planar drawing is k-gap-planar.*

Proof. let $k \in \mathbb{N}$, $k \geq 1$, let G be a $(2k)$-planar graph, and let Γ be a $(2k)$-planar drawing of G. Let $H = (A \cup B, E_H)$ be a bipartite graph obtained as follows. The set A has a vertex $a_{e,f}$ for each crossing in Γ between two edges e and f of G. For each edge e of G there are k vertices $b_e^1, \dots, b_e^k$ in B. For every pair of edges e, f of G that cross in Γ, graph H contains edges $(a_{e,f}, b_e^1), \dots, (a_{e,f}, b_e^k)$ and $(a_{e,f}, b_f^1), \dots, (a_{e,f}, b_f^k)$ in H. Notice that if H admits a matching of A in B, then each crossing of Γ between an edge e and an edge f of G can be assigned to either e or f, and no edge of G is assigned with more than k crossings.

Consider any subset A' of A, and let $B' = N(A')$ be the neighborhood of A' in B. We claim that $|A'| \leq |B'|$. Let $E' \subseteq E_H$ denote the edges between A' and B'. By construction every vertex in A has degree $2k$, and hence $|E'| \geq 2k|A'|$. On the other hand, every vertex in B has degree at most $2k$ as every edge of G has at most $2k$ crossings, and hence $|E'| \leq 2k|B'|$. Hence $|A'| \leq |B'|$ as claimed.

By Hall's theorem, it now follows that H admits a matching from A into B, which corresponds to an assignment of gaps in Γ such that no edge has more than k gaps, i.e., Γ is a k-gap-planar drawing. □

To conclude the proof of Theorem 16, we should prove that for every $k \geq 1$, there is a k-gap-planar graph that is not $(2k)$-planar. A stronger result holds:

Lemma 20. *For every $k \geq 1$, there exists a 1-gap-planar graph G_k that is not k-planar.*

Proof. Let $k \in \mathbb{N}$. We construct a graph $G_k = (V, E)$ together with its 1-gap-planar drawing as follows. Start with an edge (a, b) crossed by $k+1$ disjoint edges (c_i, d_i), for $i = 1, \dots, k+1$. The $2k+2$ vertices lie in a common face, and we can connect them by a Jordan curve, which forms a cycle $C = (a, c_1, \dots, c_{k+1}, b, d_{k+1}, \dots, d_1)$. Add a new vertex v_0 in the exterior of the cycle, and connect it to all vertices of C. The cycle C and v_0 form the wheel W, which has $m = 4k+4$ edges. Finally, replace each edge of the wheel by t edge-disjoint paths of length 2, where $t \geq k$ is a suitable parameter that we shall specify shortly. This completes the construction of $G_k = (V, E)$. Note that the total number of edges is bounded above by

$$|E| = 1 + (k+1) + (4k+4)2t = 1 + (k+1)(8t+1) < 10(k+1)t.$$

It is clear that G_k is 1-gap-planar, since the crossing between (a, b) and (c_i, d_i) can be charged to (c_i, d_i) for all $i = 1, \dots, k+1$.

Suppose that G_k admits a k-planar drawing Γ. Since each edge crosses at most k other edges, the total number of crossings is at most $k|E|/2 < 5k(k+1)t$.

We claim that for each edge of the wheel W, we can choose $k+1$ of the t paths such that no two chosen paths that correspond to different edges of the wheel cross in the drawing Γ. We prove the claim by contradiction. Since we choose $k+1$ out of t paths for each of the m edges of the wheel independently, there are $\binom{t}{k+1}^m$ possible choices. Suppose, for the sake of contradiction, that every choice produces a graph that has at least one crossing in Γ between paths corresponding to different edges of W. Each crossing between two such paths is counted $\binom{t-1}{k}^2 \binom{t}{k+1}^{m-2}$ times. Consequently, the total number of crossings in Γ is at least $t^2/(k+1)^2$. If we put $t = 5(k+1)^4$, then we would have at least $t^2/(k+1)^2 = t \cdot 5(k+1)^4/(k+1)2 = 5(k+1)^2 t > 5k(k+1)t$ crossings, a contradiction. This completes the proof of the claim.

Let G'_k be a subgraph of G_k that consists of $k+1$ paths corresponding to each edge of W such that the paths corresponding to different edges of W do not cross in Γ; and let Γ' be the restriction of Γ to G'_k. Note that any $k+1$ paths that correspond to the same edge of W are homotopic to each other in Γ'. If we pick one of the $k+1$ paths, for each edge of W, the Jordan arc along these paths provide a planar drawing of W. Since W is 3-connected, it has a combinatorially unique embedding, which we denote by $\Gamma(W)$. As noted above, every edge of W in the drawing $\Gamma(W)$ is homotopic to $k+1$ paths in the drawing Γ'.

As the combinatorial embedding of W is unique, the vertices a, b, and c_i, d_i, for $i = 1, \dots, k+1$, lie on the boundary of a single face, which we denote by F. If edges (a, b) and (c_i, d_i), for $i = 1, \dots, k+1$, are homotopic to Jordan arcs that lie in F, then (a, b) crosses (c_i, d_i), for all $i = 1, \dots, k+1$. If any of these edges is not homotopic to a Jordan arc in F, then it crosses a bundle of $k+1$ paths corresponding to some edge of W. In both cases, one of the edges crosses $k+1$ other edges in Γ, contradicting our assumption that Γ is a k-planar drawing. □

Relation to d-degenerate crossing graphs. Eppstein and Gupta [25] defined d-degenerate crossing graphs, for $d \in \mathbb{N}$. A graph is a *d-degenerate crossing graph* if it admits a drawing Γ such that the crossing graph $C(\Gamma)$ is d-degenerate. Recall that a graph is *d-degenerate* if the vertices admit a total order in which each vertex is adjacent to at most d previous vertices. It is clear from the definition that for every $k \in \mathbb{N}$, every k-degenerate crossing graph is a k-gap-planar graph. However, the converse is false for $k = 1$: We show below (Lemma 21) that for every 1-gap-planar drawing of a 1-gap-planar graph with $n \geq 20$ vertices and the maximum number of edges, the crossing graph contains a cycle, hence it is not 1-degenerate.

Lemma 21. *For every 1-gap-planar graph G with $n \geq 20$ vertices and $5n - 10$ edges and for every 1-gap-planar drawing Γ of G, the crossing graph $C(\Gamma)$ contains a cycle. Consequently, $C(\Gamma)$ is not 1-degenerate, and G is not a 1-degenerate crossing graph.*

Proof. Let $G = (V, E)$ be a 1-gap-planar graph with $n \geq 20$ vertices and $5n-10$ edges (infinite families of such graphs have been constructed in Section 3.2). Let Γ be a 1-gap-planar drawing of G, and let $C(\Gamma) = (E, X)$ be its crossing graph.

Let $H = (V, E')$ be a subgraph of G, where $E' \subseteq E$ is a maximum set of edges that are pairwise noncrossing in Γ. By Lemma 5, H is a triangulation, consequently $|E'| = 3n - 6$. Let $E'' = E \setminus E'$. The charging scheme in the proof of Theorem 6 gives a one-to-one correspondence between E'' and the $2n - 4$ faces of H such that each edge $e \in E''$ corresponds to an end triangle of e. Since G is a simple graph, the two end portions of every edge $e \in E''$ lie in two different (triangular) faces of H.

Starting with an arbitrary edge $e_1 \in E''$, we construct a sequence P of edges in E'' as follows. Assume that the edge e_i is already defined for $i \in \mathbb{N}$. Edge e_i has two distinct end triangles, and the charging scheme matches e_i to only one of them. Let Δ_i be the end triangle of e_i that is not charged by e_i, and let $e_{i+1} \in E''$ be the edge charged to Δ_i. Since E'' is finite, the sequence P contains a cyclic sequence without repetition that we denote by C_0.

We construct a cyclic sequence C_1 from C_0 that forms a cycle in the crossing graph $C(\Gamma)$ as follows. Consider two consecutive elements of C_0, say e_i and e_{i+1}, both have an end portion in triangle Δ_i. For every two consecutive edges in C_0, say e_i and e_{i+1}, do the following: If the end portions of e_i and e_{i+1} that lie in Δ_i are incident to the same vertex of Δ_i, then both edges cross the opposite side of Δ_i that we denote by f_i, and we insert edge $f_i \in E'$ into C_0 between e_i and e_{i+1}. Otherwise end portions of e_i and e_{i+1} in Δ_i are incident to two distinct vertices of Δ_i, consequently e_i and e_{i+1} cross in the interior of Δ_i, and we do not insert anything between e_i and e_{i+1}. We obtain a cyclic sequence C_1 of edges in E such that every two consecutive edges cross in the drawing Γ. We have shown that $C(\Gamma)$ contains a cycle, as claimed. ☐

Note: David Wood (private communication) has pointed out that a weak version of the converse is also true. Specifically, every k-gap-planar graph G is $2k$-degenerate. This follows from the fact that the crossing graph C of a k-gap-planar drawing of G has average degree at most $2k$.

5 1-gap-planar drawings of complete graphs

In this section, we characterize which complete graphs are 1-gap-planar.

Theorem 22. *The complete graph K_n is 1-gap-planar if and only if $n \leq 8$.*

Proof. Figure 7(a) shows a 1-gap-planar drawing of K_8, and by monotonicity the graphs $K_1, \ldots, K_7$ are 1-gap-planar as well. We now prove that K_9 is not 1-gap-planar, which again by monotonicity settles all cases K_n for $n \geq 9$.

Since K_9 has 36 edges and $\text{cr}(K_9) = 36$ [32], a 1-gap-planar drawing of K_9 can only arise from assigning exactly one gap to each edge in a crossing-minimal drawing of K_9 (cf. Property 1). We obtain a contradiction by showing that in every crossing-minimal drawing of K_9 some edge has no crossing at all.

Let Γ^* be the planarization of such a crossing-minimal drawing Γ. Note that Γ^* has $n^* = 45$ vertices and $m^* = 108$ edges (since it has 9 real vertices of degree 8 and 36 dummy vertices of degree 4), so by Euler's formula, the number of faces of Γ^* is $f^* = m^* - n^* + 2 = 108 - 45 + 2 = 65$. For a real vertex u of Γ^*, we

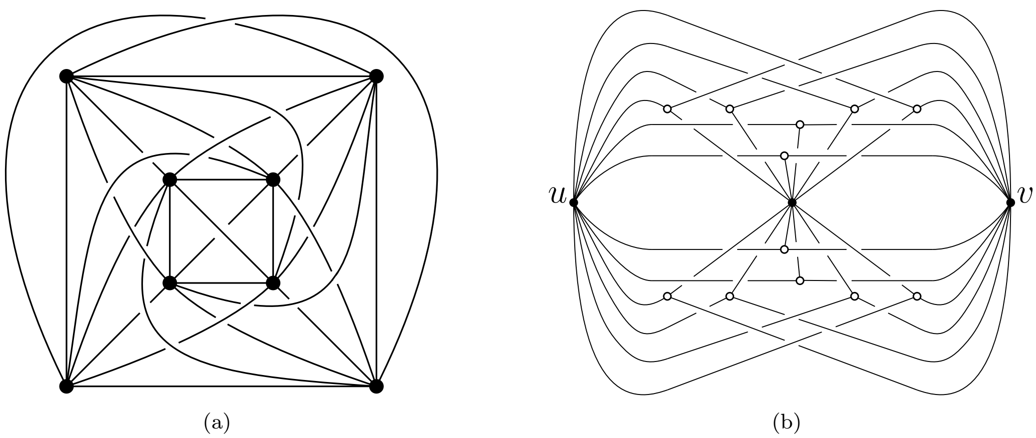


Figure 7: A 1-gap-planar drawing of (a) K_8 and (b) $K_{3,12}$.

denote by $F(u)$ the set of faces of Γ^* that are incident to u. We claim that Γ^* is biconnected and $|F(u)| = 8$ for every real vertex u of Γ^*.

Suppose, for a contradiction, that Γ^* is not biconnected. Then it contains a cut-vertex c, which is either a dummy or a real vertex. If c is a dummy vertex, note that it is adjacent to exactly two connected components of $\Gamma^* \setminus \{c\}$. Then we can reflect the drawing of one of the two components, thereby eliminating the crossing at c, which contradicts the crossing-minimality of Γ. We now show that no real vertex is a cut-vertex in Γ^*. Every 3-cycle in K_9 forms a simple cycle in Γ^* (since Γ is a simple drawing and thus adjacent edges do not cross). On the other hand, any three real vertices in Γ^* are part of a 3-cycle in K_9, and thus part of a simple cycle in Γ^*. Hence, no real vertex is a cut-vertex in Γ^*. Finally, $|F(u)| = 8$ because every real vertex u has degree 8 and Γ^* is biconnected.

It follows that there are real vertices u, v which share a face (i.e. $F(u) \cap F(v) \neq \emptyset$), as otherwise there would have to be $\sum_u |F(u)| = 9 \cdot 8 = 72 > 65 = f^*$ faces. But now the edge (u, v) can be redrawn inside this face so that this edge cannot have had any crossing to begin with since Γ was assumed to be crossing-minimal. □

6 Recognizing 1-gap-planar graphs

We denote by 1GapPlanarity the problem of deciding whether a given graph G is 1-gap-planar. We show that 1GapPlanarity is NP-complete, by a reduction from 3Partition. Recall that an instance of 3Partition consists of a multiset $A = \{a_1, a_2, \ldots, a_{3m}\}$ of $3m$ positive integers in the range $(I/4, I/2)$, where I is an integer such that $I = 1/m \cdot \sum_{i=1}^{3m} a_i$, and asks whether A can be partitioned into m subsets $A_1, A_2, \ldots, A_m$, each of cardinality 3, such that the sum of integers in each subset is I. This problem is strongly NP-hard [29], and thus we may assume that I is bounded by a polynomial in m.
We begin by showing that 1GapPlanarity is in NP.

Lemma 23. *The problem* 1GapPlanarity *is in* NP.

Proof. Given a planarization Γ^* of a drawing Γ, we can check whether it is 1-gap-planar in polynomial time by using Property 3. A nondeterministic algorithm to generate all planarizations of a graph with k crossings, where $0 \leq k < \binom{m}{2}$, evaluates all possible k pairs of edges that cross (and the order of the crossings along the edges) with a technique similar to the one in [30]. Then it replaces crossings with dummy vertices and tests whether the resulting graph is planar, i.e., whether it is a planarization of a drawing of G, and whether it is 1-gap-planar. Hence, the problem belongs to NP. □

Our reduction is reminiscent to the reduction used in [10]. However, the proof in [10] holds only for the case in which a clockwise order of the edges around each vertex is part of the input, i.e., only if the *rotation system* of the input graph is fixed. A similar reduction is also used in [11], in which the rotation system assumption is not used. However, the gadgets in [11] have a unique embedding. We do not use the fixed rotation system assumption, nor we can easily derive a unique embedding for our gadgets, and thus have to deal with additional challenges in our proof. In what follows we define a "blob" graph that will be used to enforce an ordering among the edges adjacent to certain vertices. Consider the complete bipartite graph $K_{3,12}$, whose crossing number is 30 [41, 50]. Fig. 7(b) shows a 1-gap-planar drawing of $K_{3,12}$ with exactly 30 gaps. Note that two degree-12 vertices, u and v, are drawn on the outer face. Since $K_{3,12}$ has 36 edges, the next lemma easily follows.

Lemma 24. *Every* 1*-gap-planar drawing of* $K_{3,12}$ *has at most* 6 *gap-free edges.*

A *blob* B is a copy of $K_{3,12}$. A *gapped chain* $\mathcal{C}$ of a 1-gap-planar drawing is a closed, possibly nonsimple, curve such that any point of $\mathcal{C}$ either belongs to a gapped edge or corresponds to a vertex.

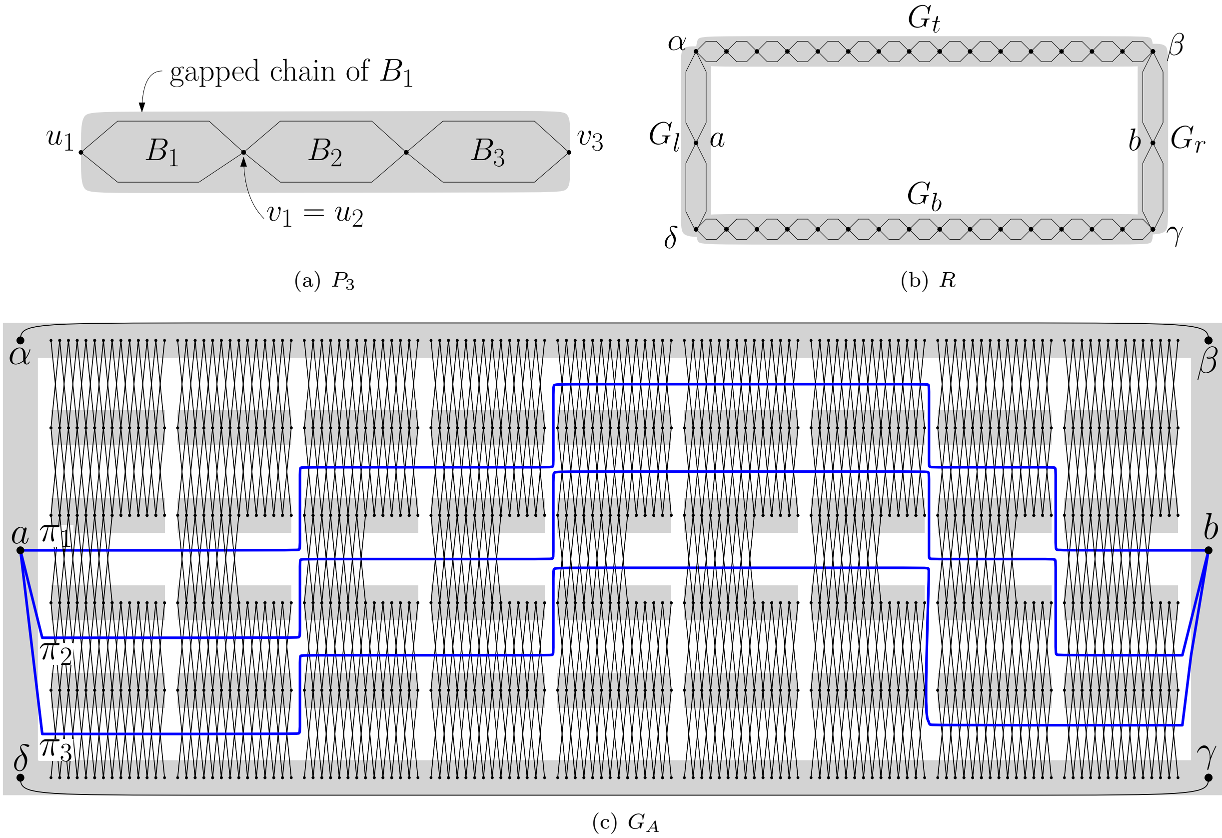
Lemma 25. *Let* u *and* v *be two degree-*12 *vertices of* B*. Every* 1*-gap-planar drawing* Γ *of* B *contains a gapped chain* $\mathcal{C}$ *containing* u *and* v*.*

Proof. Let Γ^* be the planarization of Γ. Let Γ' be the subgraph of Γ^* consisting only of those edges that correspond to or belong to gapped edges of Γ. We prove that Γ' contains two edge-disjoint paths from u to v. Note that these two edge-disjoint paths may meet at real vertices and at dummy vertices (i.e., a crossing between two gapped edges). A curve that goes through these two paths is the desired gapped chain.

According to Menger's theorem, two such paths exist if and only if every (u, v)-cut of Γ' has size at least 2, where a (u, v)-cut of Γ' is a set of edges of Γ' whose removal disconnects u and v. It is well known that such an (u, v)-cut corresponds to cycle in the dual, which in turn corresponds to a curve that separates u and v by crossing a set of edges. We now consider one such curve, and claim this curve crosses at least two gapped edges in the original drawing Γ (after a slight perturbation we can assume that it does not pass through a vertex). More precisely, let ℓ be a simple closed curve such that: it does not pass through any vertex of Γ; it divides the plane into two nonempty topologically connected regions, one containing u and the other containing v. Let L denote the set of edges of G that are crossed by ℓ. Note that G contains 12 edge-disjoint paths from u to v, which induce 12 edge-disjoint paths from u to v in Γ^*. It follows that $|L| \geq 12$. Also, Γ has at most 6 gap-free edges by Lemma 24. Hence, L contains at least $12 - 6 > 2$ gapped edges. □

We are now ready to show how to transform an instance A of 3Partition into an instance G_A of 1GapPlanarity. We start by defining some gadgets for our construction. A *path gadget* P_k is a graph obtained by merging a sequence of $k > 0$ blobs as follows. Let B_1, B_2, …, B_k, be k blobs such that u_i and v_i are two vertices of degree 12 in B_i. Identify the vertices $v_i = u_{i+1}$ for $i = 1, \ldots, k-1$, each of these vertices is called an *attaching vertex*. Thus, P_k has $k+1$ attaching vertices. In a 1-gap-planar drawing of P_k, any two gapped chains of two blobs, B_i and B_j $(i < j)$, are disjoint, except for a possible common attaching vertex. A schematization of P_k (for $k = 3$) is shown in Fig. 8(a). A *top beam*, denoted G_t, is a path gadget P_k with $k = 3m(\lceil I/2 \rceil + 2) + 1$. Recall that G_t has $3m(\lceil I/2 \rceil + 2) + 2$ attaching vertices. A *right wall* G_r is a path gadget P_k with $k = 2$. Symmetrically, a *bottom beam* G_r is a path gadget with $k = 3m(\lceil I/2 \rceil + 2) + 1$, and a *left wall* G_l is a path gadget with $k = 2$. A *global ring* R is obtained by merging G_t, G_r, G_b, and G_l in a cycle as in Fig. 8(b). Again, in any 1-gap-planar drawing Γ_R of R, the gapped chains of two distinct blobs, B_i and B_j $(i \neq j)$, are disjoint, except for a possible common attaching vertex. Thus, Γ_R contains a gapped chain C_R that is the union of all the gapped chains of the blobs of R.

We start the construction of G_A with a global ring R. Let α, β, γ, δ be the attaching vertices shared by G_l and G_t, G_t and G_r, G_r and G_b, G_b and G_l, respectively (see also Fig. 8(b)). First we add the edges (α, β) and (γ, δ). Denote as R^+ the resulting graph, and consider a 1-gap-planar drawing of this graph. The gapped chain of R subdivides the plane into a set of connected regions, such that only two of them contain all of α, β, γ, and δ on their boundaries. We denote these two regions as r_1 and r_2. For ease of illustration,



(a) P_3

(b) R

(c) G_A

Figure 8: (a) Schematization of a path gadget P_3. (b) A global ring R. (c) Schematization of the instance G_A with $m = 3$, $A = \{7, 7, 7, 8, 8, 8, 8, 9, 10\}$ and $I = 24$. Transversal paths are routed according to the following solution of 3Partition $A_1 = \{7, 7, 10\}$, $A_2 = \{7, 8, 9\}$ and $A_3 = \{8, 8, 8\}$. For simplicity, the gapped chains of the various blobs are not shown, as well as vertex w and all the degree-2 vertices of the transversal paths.

we assume that one of them is infinite (as in Fig. 8(b)), say r_2. Since the drawing is 1-gap-planar, each of (α, β) and (γ, δ) is drawn entirely in one of these two regions. We assume that both these two edges are drawn in the same region, say r_2, and we will later show that this is the only possibility in any 1-gap-planar drawing of the final graph G_A.

We continue by connecting the top and bottom beams by a set of $3m$ *columns*; refer to Fig. 8(c). We describe each column in terms of its drawing, and we will later see that this is the only possible drawing that can be part of a 1-gap-planar drawing of G_A. A column consists of $2m - 1$ *cells*; a cell consists of a set of pairs of crossing edges, called its *vertical pairs*. Cells of the same column are separated by $2m - 2$ path gadgets, called *floors*. In particular, the cell between the $(m - 1)$-st floor and the m-th floor is called the *central cell*. Again, we are assuming a particular left-to-right order for the attaching vertex of a floor, and we will see that this is the only possible order in a 1-gap-planar drawing. The central cells (we have $3m$ of them in total) have a number of vertical pairs depending on the elements of A. Specifically, the central cell C_i of the i-th column contains a_i vertical pairs connecting its delimiting floors ($i \in \{1, 2, \ldots, 3m\}$). Each of the remaining cells each has $\lceil I/2 \rceil + 1$ vertical pairs. Hence, a noncentral cell contains more edges than any central cell. Further, the number of attaching vertices of a floor can be computed based on how many vertical pairs must be connected to the gadget.

It is now straightforward to see that it is not possible to draw both a column and one of (α, β) and

(γ, δ) in r_1 or r_2 without violating 1-gap-planarity. Hence, we shall assume that both (α, β) and (γ, δ) are in r_2 and that all the columns are in r_1. Consider now a 1-gap-planar drawing of a column. If we invert the left-to-right order of the attaching vertices of a floor (i.e., we mirror its drawing), then the resulting drawing is not 1-gap-planar, since each floor delimits at least one noncentral cell with $\lceil I/2 \rceil + 1$ vertical pairs. Moreover, since each vertical pair has a gapped edge, two vertical pairs cannot cross each other in a 1-gap-planar drawing, and thus the drawings of the columns are disjoint from one another.

Finally, let a and b be the attaching vertices of the left and right walls distinct from α, β, γ, and δ. We connect a and b with m pairwise internally disjoint paths, called *transversal paths*; each transversal path has exactly $(3m-3)(\lceil I/2 \rceil + 1) + I$ edges. The routing of these paths will be used to determine a solution of A, if it exists. Thus, we aim at forcing the transversal paths to be inside r_1 in a 1-gap-planar drawing of the graph. For this purpose, adding a vertex w connected to all the attaching vertices of G_t and G_b will suffice. Due to the presence of the columns in r_1, vertex w must be in r_2 and, due to the edges (α, β) and (δ, γ) in r_2, all its incident edges (except at most two) are gapped. Thus, the transversal paths must be drawn in r_1. This concludes the construction of G_A.

Theorem 26. *The* 1GapPlanarity *problem is* NP*-complete.*

Proof. The 1GapPlanarity problem is in NP by Lemma 23.

We now prove that an instance A of 3Partition is a positive instance if and only if the graph G_A is a positive instance of 1GapPlanarity.

Suppose first that G_A is a positive instance of 1GapPlanarity. From the above discussion, it is clear that each traversing path must be routed through exactly three central cells and $3m-3$ noncentral cells. In particular, each path has $(3m-3)(\lceil I/2 \rceil + 1) + I$ edges, and hence can traverse at most these many vertical pairs. Since each noncentral cell consists of $\lceil I/2 \rceil + 1$ vertical pairs, it must be that the 3 central cells contain I vertical pairs in total. Thus, we can construct a solution for A by looking at the central cells traversed by the m paths.

Suppose now that A is a positive instance of 3Partition. Note that a 1-gap-planar drawing of G_A can be always computed if one omits all the transversal paths (see also Fig. 8(c)). To draw the paths, let $\{A_1, A_2, ..., A_m\}$ be a solution for A. Then we route the paths similarly as in [10], that is, in such a way that: (1) they do not cross each other; (2) they do not cross any blob; (3) each path passes through exactly 3 central cells with I vertical pairs in total, and $3m-3$ noncentral cells; and (4) each cell is traversed by at most one path. Consider a subset A_j of the solution of instance A of 3Partition and assume without loss of generality that $A_j = \{a_\kappa, a_\lambda, a_\mu\}$, where $1 \leq \kappa, \lambda, \mu \leq 3m$. Then, in the computed drawing, path π_j will cross the κ-th, λ-th and μ-th columns of G_A through central cells, while it will cross the remaining columns of G through noncentral cells. Hence, requirement (3) is satisfied. Consider now the routing of the remaining transversal paths through the κ-th column; the corresponding routings though the λ-th and μ-th columns of G_A are symmetric. By construction, there must exist exactly $m-1$ available cells above and exactly $m-1$ available cells below the central cell of the κ-th column. This implies that there exist at least as many available noncentral cells as transversal paths to route at each side of the central cell of the κ-th column. Hence, we can route the remaining transversal paths through the κ-th column so that all other requirements are satisfied. □

We conclude by observing that our proof can be easily adjusted for the setting in which the rotation system of the input graph is fixed. We call this problem 1GapPlanarityWithRotSys. It suffices to choose a rotation system for G_A that guarantees the existence of a 1-gap-planar drawing ignoring the transversal paths (we already discussed the details of this drawing), and such that the transversal paths are attached to a and b with the ordering of their edges around a reversed with respect to the ordering around b. The membership of the problem to NP can be easily verified (similarly to Lemma 23). Thus, the next theorem follows.

Theorem 27. *The* 1GapPlanarityWithRotSys *problem is* NP*-complete.*

Note that our proof does not distinguish between simple and nonsimple drawings: Theorems 26 and 27 work for both cases. In fact, if the rotation system is not fixed, a graph is 1-gap-planar if and only if it

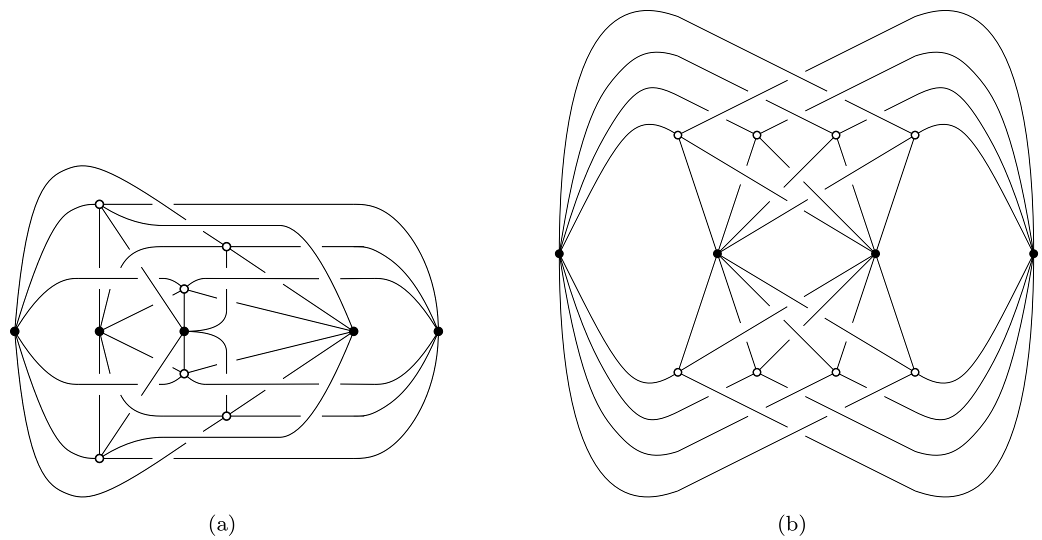


Figure 9: 1-gap-planar drawings of $K_{5,6}$ (left) and $K_{4,8}$ (right).

admits a simple 1-gap-planar drawing (self-crossings and multiple crossings can be redrawn as explained in Lemma 9). When the rotation system is fixed the statement is not always true. This is due to the fact that the redrawing may alter the rotation system. Thus, it is possible that a graph has a nonsimple 1-gap-planar drawing for some rotation system, while it does not admit a simple 1-gap-planar drawing with the same rotation system (in such a case, it must admit a simple drawing with a different rotation system).

7 Conclusions and open problems

We introduced k-gap-planar graphs, and our results give rise to several questions for future research. Among them are:

(i) In Theorem 2 we characterized k-gap-planar graphs by an adaptation of Hall's condition. We wonder whether a similar characterization is possible based on the crossing number. Specifically, does there exist some function $f : \mathbb{N} \to \mathbb{N}$ such that if $\mathrm{cr}(G') \leq f(k)|E(G')|$ for every subgraph G' of a graph G, then G is k-gap-planar?

(ii) We proved that k-gap-planar graphs with n vertices have $O(\sqrt{k} \cdot n)$ edges, which is tight apart from constant factors; and that 1-gap-planar graphs have at most $5n - 10$ edges, which is a tight bound for $n \geq 20$. Can one establish a tight bound also for 2-gap-planar graphs?

(iii) We proved that a drawing with at most $2k$ crossings per edge is k-gap-planar, and that a k-gap-planar drawing does not contain $2k + 2$ pairwise crossing edges. Do 1-gap-planar graphs have RAC drawings with at most 1 or 2 bends per edge? What is the relationship between 1-gap-planar graphs and fan-planar graphs?

(iv) We proved that K_n is 1-gap-planar if and only if $n \leq 8$. A similar characterization could be studied also for complete bipartite graphs. Note that $K_{5,7}$ is not 1-gap-planar since its crossing number is greater than its number of edges, while $K_{5,6}$ admits a 1-gap-planar drawing (Fig. 9(a)). We do not know whether $K_{6,6}$ is 1-gap-planar. Similarly, $K_{3,12}$ (Fig. 7(b)) and $K_{4,8}$ (Fig. 9(a)) are 1-gap-planar, but we do not know whether this is true also for $K_{3,13}$ and $K_{4,9}$.

(v) We proved that deciding whether a graph is 1-gap-planar is NP-complete, even if the rotation system is fixed. Can the problem be solved in polynomial time for drawings in which all vertices are on the outer boundary?

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